1.Please solve the following ordinary differential equations. (40%)

(i) $y' + 3y = 0$

ANS:

$$y'+3y=0$$

$$\frac{dy}{dx}=-3y$$

$$\int\frac{dy}{y}=\int-3dx$$

$$\ln y=-3x+c$$

$$y=e^{-3x+c}=ce^{-3x}$$

(ii) $y' = 1 + 4y^2$, $y(1) = 0$

ANS:

$$\frac{dy}{dx}=1+(2y)^2$$

$$\int\frac{dy}{1+(2y)^2}=\int dx$$

$$\frac{1}{2}\tan^{-1}(2y)=x+c$$

$$y=\frac{1}{2}\tan(2x+c)$$

$$x=1\Rightarrow y(1)=\frac{1}{2}\tan(2+c)=0\,,\,c=-2$$

$$\Rightarrow y=\frac{1}{2}\tan(2x-2)$$

(iii) $2xy\,dx + x^2\,dy = 0$

ANS:

$$\frac{du}{dx},\frac{du}{dy},u=c$$

$$2xy=\frac{\partial u}{\partial x}$$

$$u=\int 2xy\,dx+k(y)$$

$$=x^2y+k(y)$$

$$\frac{\partial u}{\partial y}=x^2y+k(y),\,k(y)=c*$$

$$u=x^2y+c*=c$$

$$x^2y=c*$$

(iv) $y' = 2y - 4x$

ANS:

$$y'-2y=-4x$$

$$p=-2,\ h=\int p(x)dx=-2x$$

$$F=e^{h}=e^{-2x},\ re^{h}=(-4x)e^{-2x}$$

$$y=e^{2x}(\int(-4x)e^{-2x}dx+c)$$

$$=e^{2x}(2xe^{-2x}+e^{-2x}+c)$$

$$=2x+1+ce^{2x}$$

(v) $y'' - 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 0$

ANS:

令 $y=e^{\lambda x}$

$$\lambda^2-4\lambda+4=0$$

$(\lambda-2)^2=0,\ \lambda=2$ 重根

$$y=(C_1+C_2x)e^{2x}$$

$$y'=2C_1e^{2x}+C_2e^{2x}+2C_2Xe^{2x}$$

$$y(0)=1=C_1$$

$$y'(0)=0=2C_1+C_2$$

$$y=(1-2x)e^{2x}$$

(vi) $x^2y'' - xy' + y = 0$, $y(1) = 1.5$, $y'(1) = 0.25$

ANS:

令 $y=x^{m},\ y'=mx^{m-1},\ y''=m(m-1)x^{m-2}$

$$m(m-1)x^{m-2}x^{2}-mx^{m-1}x+x^{m}=0$$

$$m^2-2m+1=0$$

$m=1$(重根)

$$y=(C_1+C_2\ln x)x$$

$y(1)=1.5$代入 $\Rightarrow C_1=1.5$

$$y'(x)=1.5+C_2\frac{1}{x}x+C_2\ln x$$

$y'(1)=0.25$代入 $\Rightarrow 1.5+C_2+0=0.25,\ C_2=-1.25$

$$y=(1.5-1.25\ln x)x$$

(vii) $y'' + 6y' + 9y = e^{-x}\cos 2x$, $y(0) = 0$, $y'(0) = -1$

ANS:

$$y_h = (C_1 + C_2X)e^{-3x}\ ,\ y_p = e^{-x}(K\cos 2x + M\sin 2x)$$

$$y_p' = -e^{-x}K\cos 2x - 2e^{-x}K\sin 2x - e^{-x}M\sin 2x + 2e^{-x}M\cos 2x$$

$$y_p'' = e^{-x}K\cos 2x + 2e^{-x}K\sin 2x + 2e^{-x}K\sin 2x - 4e^{-x}K\cos 2x$$
$$+ e^{-x}M\sin 2x - 2e^{-x}M\sin 2x - 2e^{-x}M\cos 2x - 4e^{-x}M\sin 2x$$

$$e^{-x}\cos 2x = K - 4K - 2M - 2M - 6K + 12M + 8K = 1 \Rightarrow 8M = 1 \Rightarrow M = \frac{1}{8}$$

$$e^{-x}\sin 2x = 2K + 2K + M - 4M - 12K - 6M + 9M = 0 \Rightarrow -8K = 0 \Rightarrow K = 0$$

$$y = y_h + y_p = C_1e^{-3x} + C_2Xe^{-3x} + \frac{1}{8}e^{-x}\sin 2x$$

$$y' = -3C_1e^{-3x} + C_2Xe^{-3x} - 3C_2Xe^{-3x} - \frac{1}{8}e^{-x}\sin 2x + \frac{1}{4}e^{-x}\cos 2x$$

$$y(0) = C_1 = 0$$

$$y'(0) = C_2 + \frac{1}{4} = -1,\ C_2 = \frac{-5}{4},\ y = \frac{-5}{4}Xe^{-3x} + \frac{1}{8}e^{-x}\sin 2x$$

(viii) $y' + xy = x/y$, $y(0) = 3$

ANS:

$$y' = \frac{x}{y} - xy$$

$$yy' = x - xy^2$$

$$y\frac{dy}{dx} = -x(y^2 - 1)$$

$$\int \frac{y}{(y^2-1)}dy = -\int x dx$$

$$\frac{1}{2}\int \frac{1}{(y^2-1)}dy^2 = -\frac{1}{2}x^2 + c$$

$$\ln\left|y^2 - 1\right| = -x^2 + c$$

$$y^2 - 1 = ce^{-x^2}$$

$$y(0) = 3 \Rightarrow 9 - 1 = c \Rightarrow c = 8$$

$$y^2 = 8e^{-x^2} + 1$$

2. (a) Please derive the differential equation for the motion of a damped free oscillation. (10%) (b) Consider an underdamping motion of a body of mass *m* = 1 kg. If the time between two consecutive maxima is 2 sec. and the maximum amplitude decreases to 1/4 of its initial value after 20 cycles, what is the force constant and damping constant of the system? (10%)

ANS

(a)

$$F = F_1 + F_2$$

$$F = ma = m\frac{d^2y}{dt^2} = my''$$

$$F_1 = -ky(\text{spring}),\ F_2 = -cy'(\text{damping force})$$

$$\Rightarrow F = F_1 + F_2\ ,my'' = -ky - cy'$$

$$\Rightarrow my'' + cy' + ky = 0$$

(b)

$$c^2 < 4mk\ \ (\text{underdamping})$$

$$B = iw^* \quad ,w^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

$$\lambda_1 = -\alpha + iw^*$$

$$\lambda_2 = -\alpha - iw^*$$

$$y(t) = e^{-\alpha t}(A\cos w^* t + B\sin w^* t)$$

$$m = 1kg\ \ ,\text{間隔兩秒},\ \ 20\ cycle$$

$$e^{-c/2 \cdot 20 \cdot 2} = \frac{1}{4} \Rightarrow e^{-20c} = 0.25$$

$$-20c = \ln 0.025 = -1.386$$

$$c = 0.069\ \ kg/s$$

$$w^* = 2\pi = \frac{1}{2m}\sqrt{4mk - c^2}$$

$$6.28 = \frac{1}{2 \cdot 1}\sqrt{4 \cdot 1 \cdot k - 0.069^2}$$

$$k \approx 39.44(kg/s^2)$$

3. Please derive the integrated rate law for the reaction and plot the concentrations of all the species as a function of time. (Initially [A] = 1.0 M, [B] = [C] = 0.0 M) (20%)

(i) A + A $\rightarrow$ B (k_1)

(ii) A $\rightarrow$ B (k_1) $\rightarrow$ C (k_2)

ANS:

(i)

$$\frac{-\mathrm{d}[\mathrm{A}]}{\mathrm{dt}} = k[\mathrm{A}]^2$$

$$-\int \frac{\mathrm{d}[\mathrm{A}]}{[\mathrm{A}]^2} = \int k\mathrm{dt} + \mathrm{c}*$$

$$\frac{-1}{[\mathrm{A}]} = -k_1 + \mathrm{c}\,,\ [\mathrm{A}]_0 = 1 \text{ 代入}, \mathrm{c} = \text{-}1$$

$$[\mathrm{A}]_t = \frac{-1}{-k_1 t - 1} = \frac{1}{k_1 t + 1}$$

$$[\mathrm{B}]_t = \mathrm{A}_0 - [\mathrm{A}]_t = \mathrm{A}_0 - \frac{1}{k_1 t + 1} = \frac{k_1 t}{k_1 t + 1}$$

(ii)

$$\frac{d[A]}{dt} = -k_1[A]$$

$$\int \frac{d[A]}{[A]} = \int -k_1 dt$$

$$\ln[A] = -k_1 t + c$$

$$[A] = e^{-k_1 t} \cdot c^*, [\mathrm{A}]_{t=0} = A_0 \Rightarrow c^*$$

$$[A]_t = A_0 e^{-k_1 t}$$

$$\frac{d[B]}{dt} = k_1[\mathrm{A}] - k_2[\mathrm{B}]$$

$$= k1 A_0 e^{-k_1 t} - k2[B] \Rightarrow \frac{d[B]}{dt} + k_2[B]_t = k_1 A_0 e^{-k_1 t}$$

$$\mathrm{P(t)} = k_2, r(t) = k_1 A_0 e^{-k_1 t}$$

$$h = \int p(t)dx = k_2 t, e^{-h} = e^{-k_2 t}$$

$$e^{h\cdot} \cdot r = e^{k_2 t} \cdot k_1 A_0 e^{-k_1 t} = k_1 A_0 e^{(k_{21} - k_1)t}$$

$$\int e^h \cdot r dx = \frac{k_1}{k_2 - k_1} A_0 e^{(k_2 - k_1)t}$$

$$[B]_t = e^{-h}\left[\int e^h \cdot r dx + c\right]$$

$$= \frac{k_1}{k_{2\_}k_1} A_0 e^{-k_1 t} + e^{-k_2 t} \cdot c$$

$$[B]_0 = 0 = \frac{k_1}{k_{2\_}k_1} A_0 + c \Rightarrow c = \frac{-k_1}{k_2 - k_1} A_0$$

$$\Rightarrow [B]_t = \frac{k_1}{k_{2\_}k_1} A_0[e^{-k_1 t} - e^{-k_2 t}]$$

$$[C]_t = A_0 - [A]_t - [B]_t$$

$$= A_0 - A_0 e^{-k_1 t} - \frac{k_1}{k_{2\_}k_1} A_0[e^{-k_1 t} - e^{-k_2 t}]$$

4. (a) Please solve the Schrödinger equation for the 1-D particle-in-a-box problem with the particle mass *m* and box length of *L*. Can the energy be zero? Why? (10%)

(b) Plot the first five energy levels and wavefunctions (10%)

ANS:

(a)

$$-\frac{\hbar^2}{2m}\psi''(x)+V(x)\psi(x)=E\psi(x)$$

$$-\frac{\hbar^2}{2m}\psi''(x)=E\psi(x)$$

$$\psi''=-(\frac{2mE}{\hbar^2})\psi=-k^2\psi$$

$$\psi=A\cos kx+B\sin kx$$

$$\psi(0)=0,\psi(L)=0$$

$$A=0,B\sin kL=0,kL=n\pi,k=\frac{n\pi}{L}$$

$$\psi=B\sin\frac{n\pi x}{L},\frac{2mE}{\hbar^2}=\frac{n^2\pi^2}{L^2}$$

$$E=\frac{n^2h^2}{8mL^2}$$

(b)

$$n = 5, E_5 = \frac{25h^2}{8mL^2}$$
$$n = 4, E_4 = \frac{16h^2}{8mL^2}$$
$$n = 3, E_3 = \frac{9h^2}{8mL^2}$$
$$n = 2, E_2 = \frac{4h^2}{8mL^2}$$
$$n = 1, E_1 = \frac{h^2}{8mL^2}$$

5. What is resonance? Show the behavior of resonance by solving an undamped forced oscillation. (10%)

ANS:

This excitation of large oscillation by matching input and natural frequency ($w=w_0$) is call resonance.

$$my''+w_0{}^2 y = 0$$
$$y_h = A\cos w_0 t + B\sin w_0 t$$
$$y''+w_0{}^2 y = \frac{F_0}{m}\cos w_0 t$$
$$y_p = t(a\cdot\cos w_0 t + b\cdot\sin w_0 t)$$
$$y_p' = t(-a\cdot w_0\sin w_0 t + b\cdot w_0\cos w_0 t) + (a\cdot\cos w_0 t + b\cdot\sin w_0 t)$$
$$y_p'' = t(-a\cdot w_0{}^2\cos w_0 t - b\cdot w_0{}^2\sin w_0 t) + 2(-a\cdot w_0\sin w_0 t + b\cdot w_0\cos w_0 t)$$
$$t(-a\cdot w_0{}^2\cos w_0 t - b\cdot w_0{}^2\sin w_0 t) + 2(-a\cdot w_0\sin w_0 t + b\cdot w_0\cos w_0 t) + w_0{}^2 t(a\cdot\cos w_0 t + b\cdot\sin w_0 t)$$
$$= \frac{F_0}{m}\cos w_0 t$$
$$-2a\cdot w_0\sin w_0 t + (-2b\cdot w_0\cos w_0 t) = \frac{F_0}{m}\cos w_0 t$$
$$-2a\cdot w_0\sin w_0 t = 0, a = 0$$
$$-2b\cdot w_0\cos w_0 t = \frac{F_0}{m}\cos w_0 t, b = -\frac{F_0}{2mw_0}$$
$$y_p = -\frac{F_0}{2mw_0} t\cdot\sin w_0 t$$

6 . If a fossilized tree is claimed to be 3200 years old, what should be its C^{14}

content expressed as a percent of the ratio C^{14} to C^{12} in a living organism. (Half-life of C^{14} = 5715 years). (10%)

ANS:

$$y = y_0 e^{-kt}$$

$$0.5 = e^{-5715k} \ , k = 1.213 * 10^{-4}$$

$$\frac{y}{y_0} = 0.678$$

$$\frac{y}{y_0} * 100\% = 67.8\% \Rightarrow \frac{C^{14}}{C^{12}} = 67.8\%$$

7. If a liter of water is vibrating up and down under the influence of gravitation in a U-shaped tube of diameter 4 cm, what is the frequency? (g = 9.8 meter/sec^2, density = 1 g/cm^3) (10%)

ANS:

$$d = 4\,\text{cm}$$

$$\text{radius} \Rightarrow r = \frac{d}{2} = \frac{4}{2} = 2\,\text{cm} = 0.02\,\text{m}$$

$$F = -F_1 \ , \ my'' = -ky \Rightarrow my'' + ky = 0$$

$$\text{restoring Force} = A\gamma(2y)$$

$$A = \pi \cdot r^2 = \pi(0.02)^2$$

$$\gamma = 9.8 * 1000 \ \ m \cdot kg \cdot s^2 / m^2 = 9800\,m \cdot nt / m^3$$

$$my'' + 2A\gamma y = 0 \Rightarrow y'' + \frac{2A\gamma}{m} y = 0 \Rightarrow y'' + w_0{}^2 = 0 \ , \ w_0{}^2 = \frac{2A\gamma}{m}$$

$$w_0{}^2 = \frac{2A\gamma}{m}$$

$$= \frac{2\pi(0.02)^2 \cdot 9800 \cdot m}{m}$$

$$= 24.63 \Rightarrow w_0 = \sqrt{24.630} = 4.963$$

$$f = \frac{w_0}{2\pi} = \frac{4.963}{2 * \pi} \approx 0.79(s^{-1})$$