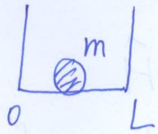
量化 Quiz 3，2015.4.13

1、Particle in a 1-d



(a)

Prove the normalization constant is $\sqrt{\frac{2}{L}}$ (30%)

$$\psi = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$

(b) Calculate $\langle P_x \rangle$ for, $n=1$ (30%)

ANS:

(a) 設 $\psi(x) = A\sin kx + B\cos kx$

邊界條件

$\psi(0)=0 \longrightarrow B\cos 0 = 0 \Rightarrow B = 0 \rightarrow \psi(x) = A\sin kx$

$\psi(L)=0 \longrightarrow \psi(L) = A\sin kL = 0 \xrightarrow{\sin kL = 0} kL = n\pi \longrightarrow k = \frac{n\pi}{L}$

$\longrightarrow \psi = A\sin\frac{n\pi x}{L}$, $\int \psi^*(x)\psi(x)\,dx = 1 \Rightarrow A^2\int_0^L \sin^2\frac{n\pi x}{L}\,dx = A^2\int_0^L \frac{1-\cos\frac{2n\pi x}{L}}{2}\,dx$

$= A^2\left[\frac{1}{2}x - \frac{1}{2}\cdot\frac{L}{2n\pi}\underbrace{\sin\left(\frac{2n\pi x}{L}\right)}_{0}\right]\Big|_0^L = 1$

$\Rightarrow A^2\cdot\frac{L}{2} = 1$，$A = \sqrt{\frac{2}{L}}$ //

(b) $\psi = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$，$P_x = -i\hbar\frac{d}{dx}$

$\langle P_x \rangle = \int_0^L \psi^* P_x \psi\,dx = \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\left(-i\hbar\frac{d}{dx}\sin\frac{\pi x}{L}\right)dx$

$\Rightarrow \frac{2}{L}\int_0^L -i\hbar\frac{\pi}{L}\left(\sin\frac{\pi x}{L}\cos\frac{\pi x}{L}\right)dx \Rightarrow \frac{-2i\hbar\pi}{L^2}\int_0^L\left(\sin\frac{\pi}{L}x\cos\frac{\pi}{L}x\right)dx \xRightarrow{\sin x\cos x = \frac{1}{2}\sin 2x}$

$\Rightarrow \frac{-i\hbar\pi}{L^2}\int_0^L \sin\frac{2\pi x}{L}\,dx = \frac{-i\hbar\pi}{L^2}\left(\frac{-L}{2\pi}\cos\frac{2\pi x}{L}\Big|_0^L\right) \xRightarrow[\cos 0 = 1]{\cos 2\pi = 1} 0$ //

2. Prove the function harmonic oscillator

$\psi = e^{-\alpha x^2/2}$, $\alpha = \sqrt{\frac{km}{\hbar^2}}$ 　　③ mm 　$V = \frac{1}{2}kx^2$ 　40%

is an eigenfunction of $\hat{H}$

and with on eigenvalue of $\frac{1}{2}h\nu$, $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

ANS:

$$H\psi = \left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right)\psi \ , \ \psi = e^{-\alpha x^2/2}$$

$$= \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}e^{-\alpha x^2/2} + \frac{1}{2}kx^2 e^{-\alpha x^2/2}$$

$$= \frac{-\hbar^2}{2m}\left[\frac{d}{dx}\left(-\alpha x e^{-\alpha x^2/2}\right)\right] + \frac{1}{2}kx^2 e^{-\alpha x^2/2}$$

$$= \frac{-\hbar^2}{2m}\left[-\alpha e^{-\alpha x^2/2} + \alpha^2 x^2 e^{-\alpha x^2/2}\right] + \frac{1}{2}kx^2 e^{-\alpha x^2/2}$$

$$= e^{-\alpha x^2/2}\left(\frac{\hbar^2\alpha}{2m} - \frac{\hbar^2\alpha^2 x^2}{2m} + \frac{1}{2}kx^2\right)$$

$$\xrightarrow{\alpha = \sqrt{\frac{km}{\hbar^2}}} e^{-\alpha x^2/2}\left(\frac{\hbar^2}{2m}\sqrt{\frac{km}{\hbar^2}} - \cancel{\frac{\hbar^2 x^2}{2m}\left(\frac{km}{\hbar^2}\right)} + \cancel{\frac{1}{2}kx^2}\right)$$

$$= e^{-\alpha x^2/2}\left(\frac{\hbar}{2}\sqrt{\frac{k}{m}}\right) = \frac{1}{2}h\nu e^{-\alpha x^2/2}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$h\nu = \frac{h}{2\pi}\sqrt{\frac{k}{m}} = \hbar\sqrt{\frac{k}{m}}$$