

1. Find a real general solution of the following system.

$$y'_1 = 6y_1 + 9y_2$$

$$y'_2 = y_1 + 6y_2$$

Ans:

$$A = \begin{bmatrix} 6 & 9 \\ 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 6-\lambda & 9 \\ 1 & 6-\lambda \end{bmatrix}$$

$$(6-\lambda)(6-\lambda) - 9 = 0 \quad \lambda = 3, 9$$

$\lambda = 3$, eigenvector :

$$3y_1 + 9y_2 = 0$$

$$y_1 + 3y_2 = 0$$

$$y_1 = -3y_2 \quad y_1/3 = -y_2$$

$$y_1 = 3 \quad y_2 = -1$$

$$X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$\lambda = 9$, eigenvector :

$$-3y_1 + 9y_2 = 0$$

$$y_1 - 3y_2 = 0$$

$$y_1 = 3y_2 \quad y_1/3 = y_2$$

$$y_1 = 3 \quad y_2 = 1$$

$$\mathbf{X} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{9t}$$

2. What kind of conic section (or pair of straight lines) is given by the quadratic form? Transform it to principal axes.

Express $\mathbf{x}^T = [x_1, x_2]$ in terms of the new coordinate vector $\mathbf{y}^T = [y_1, y_2]$.

$$4x_1^2 + 6x_1x_2 - 4x_2^2 = 10$$

Ans:

$$C = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}, \quad (4 - \lambda)(-4 - \lambda) - 9 = 0, \lambda = \pm 5$$

$$Q = 5y_1^2 - 5y_2^2 = 10,$$

$\lambda = 5$, **eigenvector:**

$$-x_1 + 3x_2 = 0$$

$$3x_1 - 9x_2 = 0$$

$$3x_2 = x_1$$

$$x_2 = x_1 / 3$$

$$x_1 = 3 \quad x_2 = 1$$

$$X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\lambda = -5$, **eigenvector:**

$$9x_1 + 3x_2 = 0$$

$$3x_1 + x_2 = 0$$

$$3x_1 + x_2 = 0$$

$$3x_1 = -x_2$$

$$x_1 = x_2 / 3$$

$$x_1 = -1 \quad x_2 = 3$$

$$X = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$x = Xy = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} y, \text{ 雙曲線}$$

